UDC 531.36

ON STABILITY OF STOCHASTIC DIFFERENTIAL EQUATIONS

PMM Vol. 41, №3, 1977, pp. 430-435
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(Received May 10, 1976)

Systems of stochastic differential equations are investigated. Theorems on exponential stability and instability of these with respect to a part of variables [1-3] under specific conditions, and, also, on exponential stability and instability in the first approximation [4-7] are proved.

1. Exponential stability with respect to a part of variables. Let us consider the system of differential equations of perturbed motion

$$dx / dt = X[t, x, \xi(t, \omega)] \tag{1.1}$$

where $\xi(t, \omega)$ $(t \geqslant 0)$ is a measurable random process with values from E_k, $X(t, x, u)$ $(x \in E_n,\ t \geqslant 0,\ u \in E_k)$ is a measurable Borel function $X(t, 0, u) \equiv 0$ with respect to (t, x, u). We shall consider the problem of stability of unperturbed motion $x = 0$ with respect to a part of variables, to be exact, with respect to $x_1, \ldots, x_m$ $(m > 0,\ n = m + p,\ p \geqslant 0)$. In conformity with [3] we denote these variables by $y_i = x_i$ $(i = 1, \ldots, m)$, and the remaining ones by $z_j = x_{m+j}$ $(j = 1, \ldots, p)$. In these variables system (1.1) assumes the form

$$dx / dt = X[t, y, z, \xi(t, \omega)] \tag{1.2}$$

We use the following notation: $\|y\| = \sup\{|y_i|;\ i = 1, \ldots, m\}$, $\|z\| = \sup\{|z_j|;\ j = 1, \ldots, p\}$, and $\|\xi\| = \sup\{|\xi_s|;\ s = 1, \ldots, k\}$.

Let us assume that the process $\xi(t, \omega)$ and function X in (1.2) are such that system (1.2) and the inital condition $x(t_0) = x_0(\omega)$ determine in region

$$t \geqslant 0, \quad \|y\| \leqslant H = \text{const}, \ \|z\| < \infty, \ \|\xi\| < \infty \tag{1.3}$$

a new absolutely continuous random process $x(t, \omega)$ with unit probability and continuous mathematical expectation with respect to t that can be continued for $t \geqslant 0$ and satisfies the equation

$$x(t, \omega) = x_0(\omega) + \int_{t_0}^{t} X[s, x(s, \omega), \xi(s, \omega)]\, ds \tag{1.4}$$

(see, e.g., [8]). Let us assume that in region (1.3) the first m equations of systems (1.1) and (1.4) satisfy conditions

$$|X_i(t, y, z, u) - X_i(t, y, 0, u)| \leqslant \varphi(t)\|y\| \quad (i = 1, \ldots, m) \tag{1.5}$$

$$|X_i(t, y'', 0, u) - X_i(t, y', 0, u)| \leqslant L\|y'' - y'\| \quad (L = \text{const};\ i = 1, \ldots, m) \tag{1.6}$$

where $\varphi(t)$ is a continuous function when $t \geqslant 0$

Definition. The solution $x = 0$ of the system of Eqs. (1.1) is called exponentially y-stable (see [3,4]), in the mean, if it is possible to find an $\varepsilon > 0$, such that for $\langle \| y_0(\omega) \| \rangle < \varepsilon,\ t \geqslant t_0$

$$\langle [\| y(t, \omega) \|; y(t, \omega) / x_0, \xi_0] \rangle \leqslant B \langle \| y_0(\omega) \| \rangle \exp[-\alpha(t - t_0)]$$

where $\alpha > 0$ and $B \geqslant 1$ are independent of t_0 and $\langle\ \rangle$ denote mathematical expectation.

Let us consider the truncated system of equations

$$dx_i / dt = X_i([t, y, 0, \xi(t, \omega)] \quad (i = 1, \ldots, m) \tag{1.7}$$

which is obtained from the first m equations of system (1.2) when $z = 0$. We denote by $y^*(t, \omega)$ the solution of Eqs. (1.7)

$$y_i^*(t, \omega) = y_{i0}^*(\omega) + \int_{t_0}^{t} X_i[s, y^*(s, \omega), 0, \xi(s, \omega)]\, ds \quad (i = 1, \ldots, m) \tag{1.8}$$

We assume that the solution $y^*(t, \omega)$ is exponentially stable in the mean, i.e., that it is possible to find an $\varepsilon > 0$, such that any solution $y^*(t, \omega)$ of Eqs. (1.7) and (1.8.) when $\langle \| y_0^*(\omega) \| \rangle < \varepsilon,\ t \geqslant t_0$ satisfies the inequality

$$\langle [\| y^*(t, \omega) \|; y^*(t, \omega) / y_0^*, \xi_0] \rangle \leqslant B \langle \| y_0^*(\omega) \| \rangle \times \exp[-\alpha(t - t_0)] \tag{1.9}$$

where $\alpha > 0$ and $B \geqslant 1$ are independent of t_0.

Theorem 1.1. If the zero solution of system (1.7) is exponentially stable in the mean and for all $t \geqslant 0$ the inequality

$$\int_{t}^{t+T} \varphi(s)\, ds \leqslant \gamma \tag{1.10}$$

where $T > 0$ is some number, is satisfied, then for a reasonably small γ the zero solution of system (1.1) is exponentially y-stable in the mean.

Proof. Let $T = \alpha^{-1} \ln(4B)$ and $\delta = \varepsilon / 2B$ where $\varepsilon > 0$ $(\varepsilon \leqslant H)$ is an a priori specified number which satisfies inequality (1.9). Then for any solution $y^*(t, \omega)$ of differential equations (1.7), whose initial functions $y^*(t_0) = y_0^*(\omega)$ satisfy the inequality $\langle \| y_0^*(\omega) \| \rangle < \delta$, the inequality

$$\langle [\| y^*(t, \omega) \|; y^*(t, \omega) / y_0^*, \xi_0] \rangle < \varepsilon / 2$$

is satisfied for all $t \geqslant t_0$ and, moreover,

$$\langle [\| y^*(t_0 + T, \omega) \|; y^*(t_0 + T, \omega) / y_0^*, \xi_0] \rangle < \delta / 4$$

Let $x(t, \omega)$ be the solution of system (1.1), which is determined by the system of input functions $x(t_0) = x_0(\omega)$ and $\xi(t_0) = \xi_0(\omega)$ in the region

$$\langle \| y_0(\omega) \| \rangle < \delta, \quad \| z_0(\omega) \| < \infty, \quad \| \xi_0(\omega) \| < \infty$$

where δ is that chosen above. Let also $y_0^*(\omega) = y_0(\omega)$. Taking into account the inequality

$$| X_i(t, y, z, u) - X_i(t, y^*, 0, u) | \leqslant \varphi(t) \| y(t) \| + L \| y(t) - y^*(t) \| \quad (i = 1, \ldots, m)$$

which follows from conditions (1.5) and (1.6) (see [1]), and the inequality

$$\| \mathbf{y}(t) \| \leqslant \| \mathbf{y}(t) - \mathbf{y}^*(t) \| + \| \mathbf{y}^*(t) \|$$

from systems (1.4) and (1.8) we obtain

$$\| \mathbf{y}(t,\omega) - \mathbf{y}^*(t,\omega) \| \leqslant \int_{t_0}^{t} \{[L + \varphi(s)] \| \mathbf{y}(s,\omega) - \mathbf{y}^*(s,\omega) \| + \varphi(s) \| \mathbf{y}^*(s,\omega) \|\} ds \tag{1.11}$$

Using the properties of mathematical expectation [9-11] and conditions (1.9) and (1.10) we obtain

$$\langle [\| \mathbf{y}(t,\omega) - \mathbf{y}^*(t,\omega) \|;\ \mathbf{y}(t,\omega) - \mathbf{y}^*(t,\omega) / \mathbf{x}_0, \xi_0] \rangle \leqslant B\delta\gamma + \int_{t_0}^{t} [L + \varphi(s)] \langle [\| \mathbf{y}(s,\omega) - \mathbf{y}^*(s,\omega) \|;\ \mathbf{y}(s,\omega) - \mathbf{y}^*(s,\omega) / \mathbf{x}_0, \xi_0] \rangle ds$$

for $t \in [t_0,\ t_0 + T]$. Applying the Gronwall-Bellman lemma (see [12]) we obtain

$$\langle [\| \mathbf{y}(t,\omega) - \mathbf{y}^*(t,\omega) \|;\ \mathbf{y}(t,\omega) - \mathbf{y}^*(t,\omega) / \mathbf{x}_0, \xi_0] \rangle \leqslant B\delta\gamma \exp(LT + \gamma)$$

when $t_0 \leqslant t \leqslant t_0 + T$. It is always possible to select γ so that

$$\langle [\| \mathbf{y}(t,\omega) - \mathbf{y}^*(t,\omega) \|;\ \mathbf{y}(t,\omega) - \mathbf{y}^*(t,\omega) / \mathbf{x}_0, \xi_0] \rangle \leqslant \delta / 4$$

Since $\langle \| \mathbf{y}(t,\omega) \| \rangle \leqslant \langle \| \mathbf{y}(t,\omega) - \mathbf{y}^*(t,\omega) \| \rangle + \langle \| \mathbf{y}^*(t,\omega) \| \rangle$, the quantity $\mathbf{y}(t,\omega)$ satisfies the inequality $t \in [t_0, t_0 + T]$

$$\langle [\| \mathbf{y}(t,\omega) \|;\ \mathbf{y}(t,\omega) / \mathbf{x}_0, \xi_0] \rangle < \varepsilon / 2 + \varepsilon / 8 < \varepsilon$$

for all $t = t_0 + T$, and for

$$\langle [\| \mathbf{y}(t_0 + T, \omega) \|;\ \mathbf{y}(t_0 + T, \omega] / \mathbf{x}_0 \xi_0] \rangle < \delta / 4 + \delta / 4 = \delta/2$$

Then using the known procedure (see, e.g., [5]) we obtain the inequality

$$\langle [\| \mathbf{y}(t,\omega) \|;\ \mathbf{y}(t,\omega) / \mathbf{x}_0, \xi_0] \rangle \leqslant B_1 \delta \exp[-\alpha_1 (t - t_0)]$$
$$(B_1 = 4B,\ \alpha_1 = (\alpha \ln 2) / (\ln 4B))$$

i.e., the zero solution of system (1.1) or (1.2) is exponentially y-stable in the mean.

Theorem 1.1 is also valid when z in system (1.2) is a vector of infinite dimension, i.e. when system (1.1) contains a denumerable number of equations. The proof is the same.

Let us consider, as an example, the system

$$dy / dt = -y + \varphi(t)\, y \sin z \xi(t, \omega)$$
$$dz / dt = G[t, y, z, \xi(t, \omega)] \tag{1.12}$$

where $\varphi(t)$ is a continuous nonnegative function when $t \geqslant 0$, $\xi(t, \omega)$ is a nonbreaking Markovian random process, and function G_t ensures the existence, uniqueness and continuability for $t \geqslant 0$ of the solution of system (1.12) which is the random process $\mathbf{x}(t, \omega) = (y(t, \omega), z(t, \omega))$, and $G[t, 0, 0, \xi(t, \omega)] \equiv 0$. The right-hand side of the first equation satisfies conditions (1.5) and (1.6). If $\sup_{[0, \infty)} \varphi(t) \leqslant 41 \cdot 10^{-3}$. then system (1.12) satisfies all conditions of Theorem 1.1, and its zero solutions are exponentially y-stable in the mean.

2. Stability according to first approximation. Let us consider besides system (1.1) the system

$$d\mathbf{x} / dt = \mathbf{X}[t, \mathbf{x}, \xi(t, \omega)] + \mathbf{R}[t, \mathbf{x}, \xi(t, \omega)] \tag{2.1}$$

where $\mathbf{R}(t, \mathbf{x}, \mathbf{u})$ ($\mathbf{x} \in E_n$, $t \geqslant 0$, $\mathbf{u} \in E_k$) is a Borel function measurable with respect to $(t, \mathbf{x}, \mathbf{u})$, and absolutely integrable over any finite time interval. Functions ξ and η with the initial condition $\mathbf{x}(t_0) = \mathbf{x}_0(\omega)$ determine in system (1.4) the unique absolutely continuous random process with unit probability. Function $\mathbf{X}$ satisfies with respect to $\mathbf{x}$ the Lipschitz condition

$$| X_i(t, \mathbf{x}'', \mathbf{u}) - X_i(t, \mathbf{x}', \mathbf{u}) | \leqslant L \| \mathbf{x}'' - \mathbf{x}' \| \quad (i = 1, \ldots, n) \tag{2.2}$$

The process ξ and function $\mathbf{R}$ in (2.1) are such that conditions

$$| R_i(t, \mathbf{x}, \mathbf{u}) | \leqslant \varphi(t) \| x \| \quad (i = 1, \ldots n) \tag{2.3}$$

are satisfied, and $\varphi(t)$ is a continuous function for $t \geqslant 0$.

The following theorem is valid.

Theorem 2.1. If the zero solution of Eqs. (1.1) is exponentially stable in the mean and if the inequalities (2.3) and (1.10) are satisfied for all $t \geqslant 0$, then for reasonably small γ the zero solution of Eqs. (2.1) is also exponentially stable in the mean.

The proof is similar to that of Theorem 1.1 (see [5]). The same proof is used in the case of denumerable systems. Theorem 2.1 was proved in [4, 6] in different conditions by the Liapunov method.

3. Exponential instability with respect to a part of variables. Let us assume that the right-hand sides of the system of Eqs. (1.1) satisfy conditions described in Sect. 1, and that in (1.3) H is fairly considerable or $H = \infty$. We introduce the following definition.

Definition [1]. We call solution $\mathbf{x} = 0$ of the system of Eqs. (1.1) exponentially y -unstable in the mean, if

$$\langle [\| \mathbf{y}(t,\omega) \| ; \mathbf{y}(t,\omega) / \mathbf{x}_0, \xi_0] \rangle \geqslant B \langle \| y_0(\omega) \| \rangle \exp [\alpha(t-t_0)]$$

where $\alpha > 0$ and $B \in (0,1]$ are independent of t_0 and $\mathbf{x}_0(\omega)$.

Let us assume that solution $\mathbf{y}^*(t,\omega)$ of system (1.7) is exponentially unstable in the mean, i.e. that for any t_0 and $\mathbf{y}_0^*(\omega)$ solution $\mathbf{y}^*(t,\omega)$ satisfies the inequality

$$\langle [\| \mathbf{y}^*(t,\omega) \|; y^*(t,\omega) / \mathbf{y}_0^*, \xi_0] \rangle \geqslant B \langle \| \mathbf{y}_0^*(\omega) \| \rangle \exp [\alpha(t-t_0)] \tag{3.1}$$

Theorem 3.1. If the zero solution of the system of Eqs. (1.7) is exponentially unstable in the mean and (1.10) is satisfied for all $t \geqslant 0$, the zero solution of the system of Eqs. (1.1) is exponentially $\mathbf{y}$ -unstable in the mean for any reasonably small γ.

Proof. Let $T = \alpha^{-1} \ln^5/_2/B$ and $\delta = \varepsilon/B$ where ε is an arbitrary (also, arbitrarily small) positive number, and B and α are numbers that appear in (3.1). Equations (1.8) with conditions (1.6) yield the estimate

$$\| \mathbf{y}^*(t,\omega) \| \leqslant \| \mathbf{y}_0^*(\omega) \| \exp LT, \quad t \in [t_0, t_0 + T] \tag{3.2}$$

From the first m equations of system (1.4) and Eqs. (1.8) we obtain inequality (1.11). Taking into account condition (1.10) and inequality (3.2) we obtain

$$\langle \| \mathbf{y}(t) - \mathbf{y}^*(t) \| \rangle \leqslant \langle \| \mathbf{y}_0^* \| \rangle \gamma \exp LT + \int_{t_0}^{t} [L + \varphi(s)] \langle \| \mathbf{y}(s) - \mathbf{y}^*(s) \| \rangle \, ds$$

for $t_0 \leqslant t \leqslant t_0 + T$.

Applying the Gronwall-Bellman lemma we obtain

$$\langle \| \mathbf{y}(t) - \mathbf{y}^*(t) \| \rangle \leqslant \langle \| \mathbf{y}_0^* \| \rangle \gamma \exp (2LT + \gamma) \quad \text{for } t_0 \leqslant t \leqslant t_0 + T$$

We make γ satisfy condition

$$\gamma \exp (2LT + \gamma) \leqslant {}^1/_2 B$$

where B is the number appearing in (3.1). We have

$$\langle \| \mathbf{y}(t) - \mathbf{y}^*(t) \| \rangle \leqslant {}^1/_2 B \langle \| \mathbf{y}_0^* \| \rangle \quad (t \in [t_0, t_0 + T]) \tag{3.3}$$

Let $\langle \| \mathbf{y}_0^* \| \rangle \geqslant 2\delta$. We have $\langle \| \mathbf{y}^*(t) \| \rangle \geqslant B \langle \| y_0^* \| \rangle$ for $t_0 \leqslant t \leqslant t_0 + T$ and $\langle \| \mathbf{y}^*(t_0 + T) \| \rangle \geqslant {}^5/_2 \langle \| \mathbf{y}_0^* \| \rangle$. Since $\| \mathbf{y}(t) \| \geqslant \| \mathbf{y}^*(t) \| - \| \mathbf{y}(t) - \mathbf{y}^*(t) \|$, $\langle \| \mathbf{y}(t) \| \rangle \geqslant \langle \| \mathbf{y}^*(t) \| \rangle - \langle \| \mathbf{y}(t) - \mathbf{y}^*(t) \| \rangle$, hence with allowance for (3.3) we obtain

$$\langle\| \mathbf{y}(t) \|\rangle \geqslant B \langle\| \mathbf{y}_0^* \|\rangle - {}^1/_2\, B \langle\| \mathbf{y}_0^* \|\rangle \geqslant B\delta = \varepsilon$$
$$\langle\| \mathbf{y}(t) \|\rangle \geqslant \varepsilon \quad \text{for} \quad t_0 \leqslant t \leqslant t_0 + T$$
$$\langle\| \mathbf{y}(t_0 + T) \|\rangle \geqslant {}^5/_2 \langle\| \mathbf{y}_0^* \|\rangle - {}^1/_2\; B \langle\| y_0^* \|\rangle \geqslant$$
$$2 \langle\| \mathbf{y}_0^* \|\rangle \geqslant 4\delta, \quad \langle\| \mathbf{y}(t_0 + T) \|\rangle \geqslant 4\delta$$

Using the method of complete mathematical induction and assuming that

$$\langle\| \mathbf{y}(t) \|\rangle \geqslant 2^{n-1}\varepsilon \quad \text{for} \quad t_0 + (n - 1)\, T \leqslant t \leqslant t_0 + nT \tag{3.4}$$
$$\langle\| \mathbf{y}(t_0 + nT) \|\rangle \geqslant 2^{n+1}\delta$$

we shall show that

$$\langle\| \mathbf{y}(t) \|\rangle \geqslant 2^{n}\varepsilon \quad \text{for} \quad t_0 + nT \leqslant t \leqslant t_0 + (n+1)\,T \tag{3.5}$$
$$\langle\| \mathbf{y}(t_0 + (n + 1)\; T) \|\rangle \geqslant 2^{n+2}\delta \tag{3.6}$$

We take the instant of time $t_0 + nT = t_0'$, as the initial instant and consider solution $\mathbf{y}^*(t)$ with initial condition $\mathbf{y}^*(t_0') = \mathbf{y}(t_0')$. From (3.4)

$$\langle\| \mathbf{y}^*(t_0') \|\rangle = \langle\| \mathbf{y}(t_0') \|\rangle \geqslant 2^{n+1}\delta$$

and from (3.1)

$$\langle\| \mathbf{y}^*(t) \|\rangle \geqslant B \langle\| \mathbf{y}^*(t_0') \|\rangle \quad \text{for} \quad t_0' \leqslant t \leqslant t_0' + T$$
$$\langle\| \mathbf{y}^*(t_0' + T) \|\rangle \geqslant {}^5/_2 \langle\| \mathbf{y}^*(t_0') \|\rangle$$

Taking into account (3.3) we obtain

$$\langle\| \mathbf{y}(t) \|\rangle \geqslant B \langle\| \mathbf{y}^*(t_0') \|\rangle - {}^1/_2\, B \langle\| \mathbf{y}^*(t_0') \|\rangle =$$
$${}^1/_2\, B \langle\| \mathbf{y}^*(t_0') \|\rangle \geqslant 2^n\varepsilon \quad \text{for} \quad t_0' \leqslant t \leqslant t_0' + T$$
$$\langle\| \mathbf{y}(t_0' + T) \|\rangle \geqslant {}^5/_2 \; \langle\| \mathbf{y}^* \;(t'_0) \|\rangle - {}^1/_2\; B \langle\| \mathbf{y}^*(t_0') \|\rangle \geqslant$$
$$2 \langle\| \mathbf{y}^*(t_0') \|\rangle \geqslant 2^{n+2}\,\delta$$

The inequalities (3.5) and (3.6) are proved. Setting $nT \leqslant t - t_0 = nT + \theta < (n + 1)\, T$, from inequality (3.5) we deduce

$$\langle [\| \mathbf{y}(t, \omega) \|; \; \mathbf{y}(t, \omega) / \mathbf{x}_0, \xi_0] \rangle \geqslant B_1 2\delta \exp \alpha_1 (t - t_0)$$
$$B_1 = \frac{1}{4} B, \quad \alpha_1 = (\alpha \ln 2) \Big/ \left(\ln \frac{5/2}{B}\right)$$

The theorem is proved.

The proof of Theorem 3.1 holds also for denumerable systems, i.e. for infinitely dimensional z.

Example. Let φ, ξ and G in system

$$dy / dt = y + \varphi(t)\, y \sin z\xi(t, \omega) \tag{3.7}$$
$$dz / dt = G[t, y, z, \xi(t, \omega)]$$

satisfy the conditions stipulated in the example in Sect. 1. If $\sup_{(0,\infty)} \varphi(t) \leqslant 8 \cdot 10^{-2}$, system (3.7) satisfies all conditions of Theorem 3.1 and, consequently, the zero solution of the system is exponentially y -unstable in the mean.

4. Instability in the first approximation. Let us again consider systems (1.1) and (2.1) for which conditions (2.2), (2.3), and (1.10) are satisfied. In that case the following theorem is valid.

Theorem 4.1. If the zero solution of Eqs. (1.1) is exponentially unstable in the mean, the zero solution of Eqs. 2.1) is also exponentially unstable in the mean for fairly small γ.

The proof is similar to that of Theorem 3.1 (see also [7]). Theorem 4.1 is valid also for denumerable systems of the form (1.1) and (2.1).

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Translated by J.J.D.